1.1

Real Line

Learning Objectives:

- To study the representation of real numbers on the numbered line called real line
- To study the algebraic, order and completeness properties
- To study the characterization of rational and irrational numbers
- To study different types of intervals on the real line
- To introduce the concept of the absolute value of a real number and to study its properties
  AND
- To solve related problems

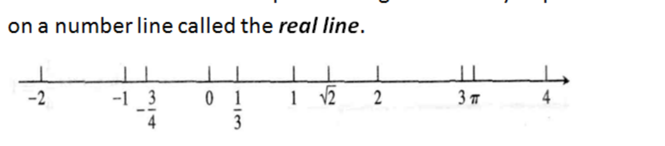
Real Numbers

Real numbers are numbers that can be expressed as decimals, such as

$$-\frac{3}{4} = -0.75000\cdots$$
$$\frac{1}{3} = 0.33333\cdots$$
$$\sqrt{2} = 1.4142\cdots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever.

The real numbers can be represented geometrically as points on a number line called the ***real line***.



The symbol $\boldsymbol{R}$ denotes either the ***real number system*** or, equivalently, the *real line*.

The properties of the real number system fall into three categories:

- **algebraic properties**
- **order properties**
  and
- **completeness**

The *algebraic properties* say that the real numbers can be added, subtracted, multiplied and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by 0.*

The *order properties* of real numbers are summarized in the following list. If a, b *and* c are real numbers, then:

1. $a<b \Rightarrow a+c<b+c$
2. $a<b \Rightarrow a-c<b-c$
3. $a<b$ and $c>0 \Rightarrow ac<bc$
4. $a<b$ and $c<0 \Rightarrow ac>bc$; $\quad a<b \Rightarrow -a>-b$
5. $a>0 \Rightarrow \frac{1}{a}>0$
6. If a *and* b are both positive or both negative, then
   $a<b \Rightarrow \frac{1}{a}>\frac{1}{b}$

The *completeness property* of the real number system says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it.

Three special subsets of real numbers are:

1. The ***natural numbers***, namely 1, 2, 3, 4, ...
2. The ***integers***, namely ... −3, −2, −1, 0, 1, 2, 3, ...
3. The ***rational numbers***, namely the numbers that can be expressed in the form of a fraction $\frac{m}{n}$, where m and n are integers and $n \neq 0$. Examples are $\frac{1}{3}, -\frac{4}{9}, \frac{200}{13}, 57=\frac{57}{1}$

The rational numbers are precisely the real numbers with decimal expansions that are either

a) *terminating* (ending in an infinite string of zeros), for example, $\frac{3}{4}=0.75000\cdots=0.75$ or

b) *repeating* (ending with a block of digits that repeats over and over), for example, $\frac{23}{11}=2.090909\cdots=2.0\overline{9}$. The bar indicates the block of repeating digits.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, *there is no rational number whose square is 2;* there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called ***irrational numbers***. They are characterized by having *nonterminating* and *nonrepeating decimal expansions*. Examples are $\pi, \sqrt{2}, \sqrt[3]{5}$, and $\log_{10} 3$

The sets of natural numbers, integers and rational numbers are respectively denoted by $\boldsymbol{N}$, $\boldsymbol{Z}$ and Q.

Intervals

A subset of a real line is called an **interval** if it contains all the real numbers lying between two points, called the *end points*. Geometrically, *intervals correspond to line segments and rays on the real line*. Intervals of real numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its end points, **half-open** if it contains one endpoint, and **open** if it contains neither endpoint. If *a* < *b*, then the **open interval** from *a* to *b*, denoted by (*a*, *b*), is the line segment extending from *a* to *b*, *excluding* the endpoints; and the **closed interval** from *a* to *b*, denoted by [*a*, *b*], is the line segment extending from *a* to *b*, *including* the endpoints. These sets can be expressed in set-builder notation as

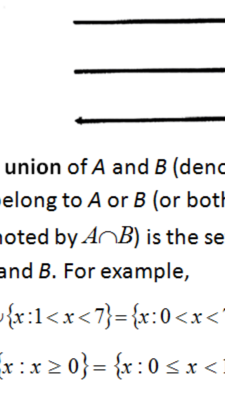
$$(a,b)=\{x : a<x<b\}$$
$$[a,b]=[x : a \le x \le b]$$

The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points**, and together make up the **interior** of the interval.

An interval may extend indefinitely in one or both directions. To indicate that an interval extends indefinitely in the positive direction we write $+\infty$ in place of a right endpoint, and to indicate that an interval extends indefinitely in the negative direction we write $-\infty$ in place of a left endpoint. Intervals that extend between two real numbers are called *finite intervals*, whereas intervals that extend indefinitely in one or both directions are called *infinite intervals*.

Parentheses and open dots mark endpoints that are excluded from the interval, whereas brackets and closed dots mark endpoints that are included in the interval. The various types of intervals are depicted below.

(a,b)	$\{x \mid a<x<b\}$
$[a,b]$	$\{x \mid a \le x \le b\}$
$[a,b)$	$\{x \mid a \le x < b\}$
$(a,b]$	$\{x \mid a < x \le b\}$
(a,∞)	$\{x \mid x > a\}$
$[a,\infty)$	$\{x \mid x \ge a\}$
$(-\infty,b)$	$\{x \mid x < b\}$
$(-\infty,b]$	$\{x \mid x \le b\}$
$(-\infty,\infty)$	**R** (set of all real numbers)



If *A* and *B* are sets, then the **union** of *A* and *B* (denoted by $A \cup B$) is the set whose members belong to *A* or *B* (or both), and the **intersection** of *A* and *B* (denoted by $A \cap B$) is the set whose members belong to both *A* and *B*. For example,

$$\{x:0<x<5\} \cup \{x:1<x<7\} = \{x:0<x<7\}$$
$$\{x:x<1\} \cap \{x:x \ge 0\} = \{x:0 \le x<1\}$$
$$\{x:x<0\} \cap \{x:x>0\} = \Phi$$

In interval notation, these sets are denoted by

$$(0,5) \cup (1,7) = (0,7)$$
$$(-\infty,1) \cap [0,+\infty) = [0,1)$$
$$(-\infty,0) \cap (0,+\infty) = \Phi$$

Absolute Value

The absolute value or magnitude of a real number *x*, denoted by |*x*|, is defined by the formula

$$|x| = \begin{cases} x & , \text{ if } x \ge 0 \\ -x & , \text{ if } x<0 \end{cases}$$

Example: |5| = 5, $\left|-\frac{4}{7}\right| = -\left(-\frac{4}{7}\right) = \frac{4}{7}$, |0| = 0

The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative. We note that

$|x| \ge 0$ for every real number *x*, and $|x|=0$ if and only if $x=0$.

The absolute value has the following properties.

1. $|-a|=|a|$
2. $|ab|=|a||b|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$
4. $|a+b| \le |a|+|b|$ (The triangle inequality)

If *a* and *b* differ in sign, then |*a* + *b*| is less than |*a*| + |*b*|. In all other cases, |*a* + *b*| equals |*a*| + |*b*|.

Example: $|-3+5|=|2|=2<|-3|+|5|=8$

$$|3+5|=|8|=|3|+|5|$$

$$|-3-5|=|-8|=8=|-3|+|-5|$$

Example

Solve the equation $|2x-3|=7$

Depending on whether $2x-3$ is positive or negative; the equation $|2x-3|=7$ can be written as

$2x-3=7$ *and* $2x-3=-7$

Solving these two equations gives

$2x=10$ *and* $2x=-4$

$x=5$ *and* $x=-2$

The solutions are $x=5$ and $x=-2$

A real number is called a **square root** of a if its square is a. Every positive real number has two square roots, one positive and one negative; the positive square root is denoted by $\sqrt{a}$ and the negative square root by $-\sqrt{a}$. For example, the positive square root of 9 is $\sqrt{9}=3$, and the negative square root of 9 is $-\sqrt{9}=-3$.

In functional context, the square root, $\sqrt{a}$ is usually defined as the nonnegative square root of a.

In such a context, it is an error to replace $\sqrt{a^2}$ by a. Although this is correct when a is nonnegative, it is false for negative a. For example, if $a=-4$, then

$$\sqrt{a^2}=\sqrt{(-4)^2}=\sqrt{16}=4 \neq a$$

A result that is correct for all a is that for any real number a $\sqrt{a^2}=|a|$. Do not write $\sqrt{a^2}=a$ unless you already know that $a \ge 0$.

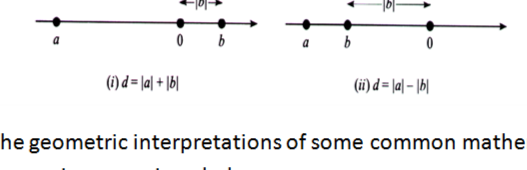
Since the symbol $\sqrt{a}$ always denotes the nonnegative square root of a, an alternate definition of $|x|$ is $|x|=\sqrt{x^2}$.

The absolute value of a real number a may be viewed as the distance between a and the origin O on the real line. The distance between two points a and b is obtained from the formula $d(a,b)=|b-a|$

This means:

$$d=d(a,b)=\begin{cases} |a|+|b|, & \text{if } a \text{ and } b \text{ have different signs} \\ |a|-|b|, & \text{if } a \text{ and } b \text{ have the same sign and } |a| \ge |b| \end{cases}$$

These two cases are pictured below.



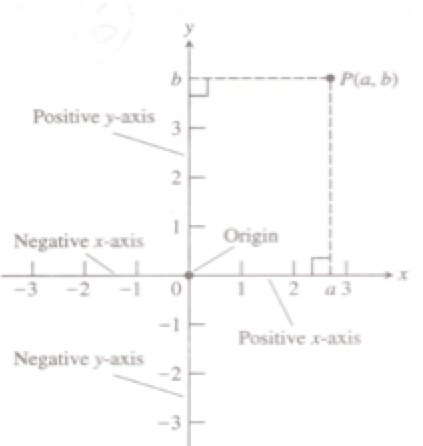
(i) $d=|a|+|b|$ (ii) $d=|a|-|b|$

The geometric interpretations of some common mathematical expressions are given below.

Expression	Geometric Interpretation on Real Line
$\lvert x-a \rvert$	The distance between x and a.
$\lvert x+a \rvert$	The distance between x and $-a$ (since $\lvert x+a \rvert = \lvert x-(-a) \rvert$)
$\lvert x \rvert$	The distance between x and the origin (since $\lvert x \rvert = \lvert x-0 \rvert$)

1.2
Cartesian Coordinates

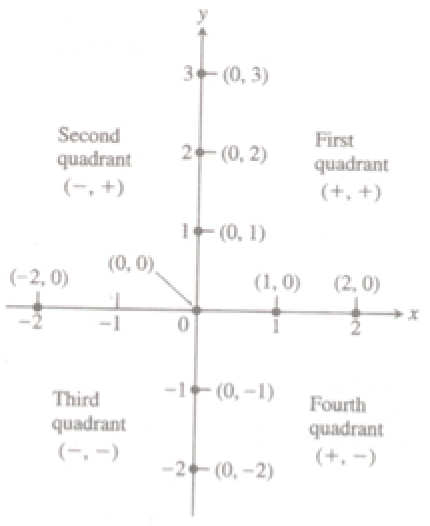
A ***rectangular coordinate system*** (also called a ***Cartesian coordinate system***) consists of two perpendicular lines (Real lines), called ***coordinate axes***, which intersect at their origins. The intersection of the axes is called the ***origin*** of the coordinate system. It is common to call the horizontal axis the x-axis and the vertical axis the y-axis, and the plane and the axes together are referred to as the xy-plane. The position of all points in the plane can be measured with respect to these two axes. On the horizontal x-axis, (Real) numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upwards. The point where x and y are both 0 is the origin of the coordinate system, denoted by the letter O.



Just as points on a coordinate line can be associated with real numbers, so points in a plane can be associated with pairs of real numbers using the rectangular coordinate system.

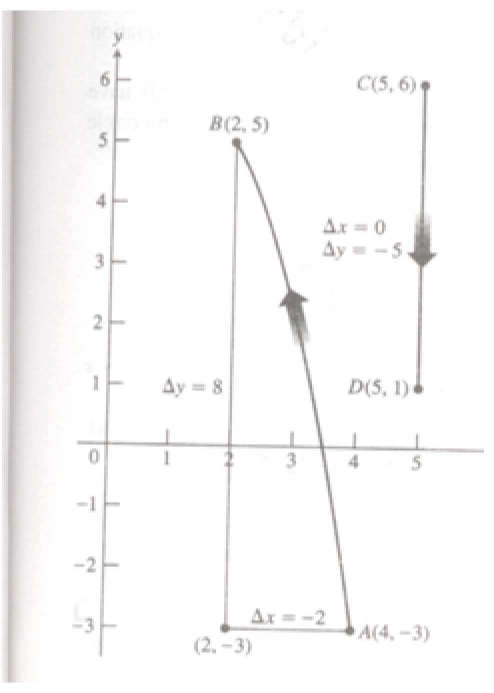
If P is any point in the plane, we can draw lines through P perpendicular to the two coordinate axes. If the lines meet the x-axis at *a* and the y-axis at *b*, then *a* is the **x-coordinate** (also called ***abscissa***) of P, and *b* is the **y-coordinate** (also called ***ordinate***). The *ordered pair* (*a*, *b*) is the point's **coordinate pair** or **the (Cartesian) coordinates of the point P**. The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point (0, 0).

The origin divides the x-axis into the **positive x-axis** to the right and the **negative x-axis** to the left. It divides the y-axis into the **positive** and **negative y-axis** above and below. The axes divide the plane into four regions called **quadrants**, numbered counter clockwise.



When a particle moves from one point in the plane to another, the net changes in its coordinates are called ***increments***. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

An increment in a variable is a net change in that variable. If x changes from x_1 to x_2, the increment in x is $\Delta x = x_2 - x_1$



In going from the point A (4, -3) to the point B (2, 5), the increments in the *x*- and *y*- coordinates are

$$\Delta x = 2 - 4 = -2 \ , \ \Delta y = 5 - (-3) = 8$$

From C (5, 6) to D (5, 1), the coordinate increments are

$$\Delta x = 5 - 5 = 0$$

$$\Delta y = 1 - 6 = -5$$

1.3

Functions

Learning objectives:

- To define a function and its domain and range
- To determine the domain and range of real valued functions of a real variable.
- To define the Sums, Differences, Products and Quotients of functions and determine their domains.
- To learn the concepts of composite functions, even and odd functions and piecewise defined functions.
  And
- To solve related problems.

Functions are used to describe the relationships between variable quantities and hence play a central role in applications. For example, an engineer may need to know how the illumination from a light source on an object is related to the distance between the object and the source.

Suppose the value of one variable quantity, called y, depends on the value of another variable quantity, called x. If the value of y is completely determined by the value of x, then we say that

y is a function of x.

If A is the area and r is the radius of a circle then we have $A = \pi r^2$. Thus A is a function of r. Now, the equation $A = \pi r^2$ is a *rule* that tells how to calculate a unique output value of A for each possible input value of the radius r.

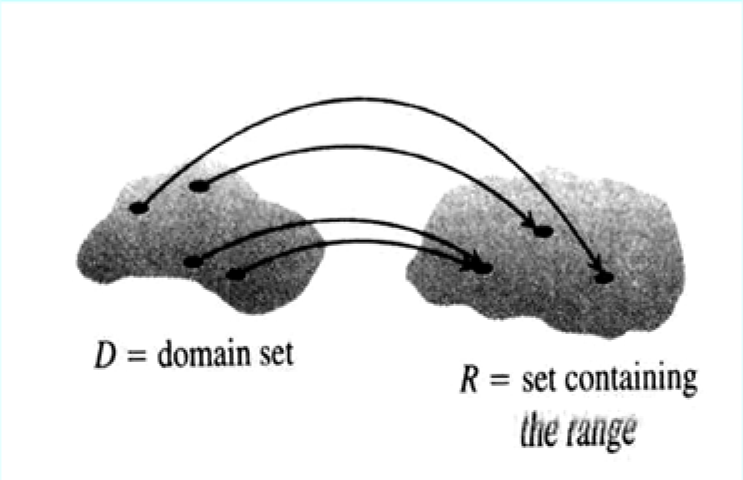
The set of all possible input values for x is the ***Domain*** of the function. The set of all output values of y is the ***Range*** of the function.

Since the circles cannot have negative radii or areas, the domain and the range of these are both in the interval $[0, \infty)$, consisting of all nonnegative real numbers.

We often refer to a generic function without having any particular formula in mind. Euler, a Swiss mathematician, gave a symbolic way to say "***y is a function of x***" by writing

$$y = f(x) \quad (\text{"}y \text{ equals } f \text{ of } x\text{"})$$

In this notation, the symbol f represents the function. The letter x, called the **Independent variable**, represents an input value from the domain of f, and y, the **dependent variable**, represents the corresponding output value of $f(x)$ in the range of f.



Thus, a function is usually expressed in one of two ways:

1. By giving formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function, or
2. By giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

We use the symbol $f(x)$ both for representing the function and denoting the value of the function at the point x. It is also convenient to use a single letter to denote both a function and its dependent variable. For instance, we might say that the area A of a circle of radius r is given by the function $A(r) = \pi r^2$.

Example

The volume V of a ball (solid sphere) of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3$$

The volume of a ball of radius 3 m is

$$V(3) = \frac{4}{3}\pi (3)^3 = 36\pi \ m^3$$

Example

Suppose that the function f is defined for all real numbers t by the formula

$$f(t) = 2(t-1) + 3$$

Evaluate f at the input values $0, 2, x+2$, and $f(2)$.

Solution

$$f(0) = 2(0-1) + 3 = 1$$
$$f(2) = 2(2-1) + 3 = 5$$
$$f(x+2) = 2(x+2-1) + 3 = 2x + 5$$
$$f(f(2)) = f(5) = 2(5-1) + 3 = 11$$

1.4

Graphs of Functions

Learning Objectives

- To learn the concepts of solution, solution set and the graph of an equation in variables x and y
- To define x and y intercepts of a graph and to learn vertical line test for the graph of a function
- To study the graphs of
  i. Absolute value function
  ii. Greatest and least integer functions
  iii. Power functions and
  iv. Circles and Parabolas
- To study the concept of shifting a graph

Graphs

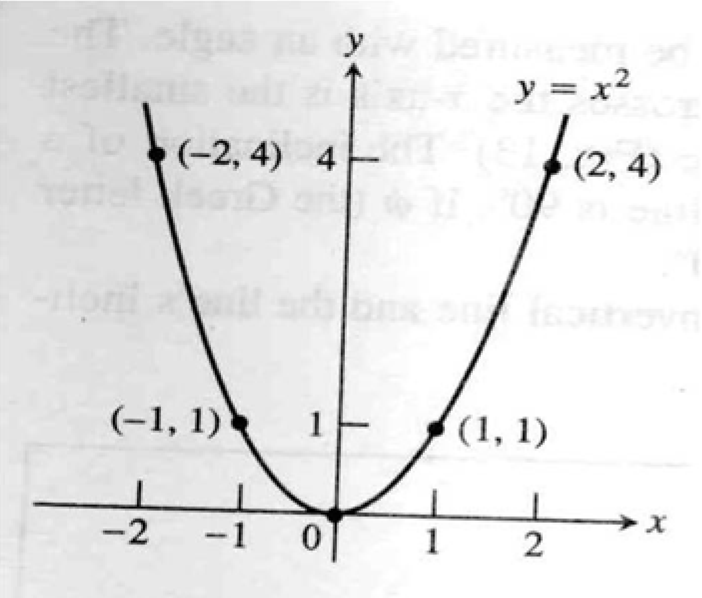
The correspondence between points in a plane and ordered pairs of real numbers will enable the visualization of algebraic equations as geometric curves, and, conversely, to represent geometric curves by algebraic equations.

Suppose that we have a xy-coordinate system and an equation involving two variables x and y, say $6x - 4y = 10$. We define a ***solution*** of such an equation to be any ordered pair of real numbers (a, b) whose coordinates satisfy the equation when we substitute $x = a$ and $y = b$. For example, the ordered pair $(3, 2)$ is a solution of the equation $6x - 4y = 10$, since the equation is satisfied by $x = 3$ and $y = 2$. However, the ordered pair $(2, 0)$ is not a solution of this equation, since the equation is not satisfied by $x = 2$ and $y = 0$.

A solution of an equation involving two variables x and y is an ordered pair of real numbers (a, b) whose coordinates satisfy the equation when a, b are substituted for x and y respectively.

The set of all solutions of an equation in x and y is called the solution set of the equation.

The graph of an equation or inequality involving the variables x and y is the set of all points $P(x, y)$ whose coordinates satisfy the equation or inequality.



The parabola $y = x^2$.

The above figure is the graph of the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0,0)$, $(1,1)$, $(-1,1)$, $(2,4)$, and $(-2,4)$. These points, and all others satisfying the equation, make up a smooth curve called a ***parabola***.

A graph intersects the x-axis at a point which has the form $(a, 0)$ and the y-axis at a point which has the form $(0, b)$. The number a is called an ***x-intercept*** of the graph and the number b is called an ***y-intercept***.

The vertical line Test:

A function f can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once. If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.

The graph of a function f is the graph of the equation $y = f(x)$. It consists of the points in the plane whose coordinates (x, y) are input-output pairs for f. The graph of a function can be obtained by plotting several coordinate pairs that satisfy the functional rule and joining them by a smooth curve.

1.5

Exponential Functions

Learning Objectives

- To define an exponential function and to study its graph called exponential curve
  And
- To practice problems on compound interest and half life of a radioactive substance.

Exponential Functions

From the theory of indices, we have the following relations

$$a^m = a.a.a \dots .a(m\ times), a^0 = 1, a^{-m} = \frac{1}{a^m}$$

where m is a positive integer.

Exponents are extended to include all rational numbers by defining

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

for any rational number $\frac{m}{n}$.

For example, $2^4 = 16, 2^{-4} = \frac{1}{16}, (125)^{\frac{2}{3}} = 5^2 = 25$

Example 1 : Evaluate $2^5, 3^{-4}, (8)^{\frac{2}{3}}, (25)^{\frac{3}{2}}$

Solution:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

$$3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

$$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$$

$$25^{-\frac{3}{2}} = \frac{1}{25^{\frac{3}{2}}} = \frac{1}{\left(\sqrt{25}\right)^3} = \frac{1}{5^3} = \frac{1}{125}$$

As a further extension, the exponents can also be allowed to be real numbers. We may define an exponential function as follows.

A function of the form $f(x) = b^x$, where the base b is a positive constant, $b \neq 1$ is called an exponential function.
The domain and range of the exponential function are $(-\infty, \infty)$ and $(0, \infty)$ respectively. An exponential function never assumes the value 0.
Each of the following is an exponential function:

$$f(x) = 2^x, y = 3^x, f(x) = \left(\frac{1}{4}\right)^x$$

Example 2: The decay of radioactive iodine-131 is described by the exponential function

$$A = A_0 2^{(-t/8)}$$

where A and A_0 are measured in μg and t in days.
Find it s half-life. The half life of a decaying substance is defined as the time it takes to decrease to half of its original amount.

Solution

Half life is given by

$$\frac{A_0}{2} = A_0 \cdot 2^{-t/8}$$

$$-1 = -\frac{t}{8}$$

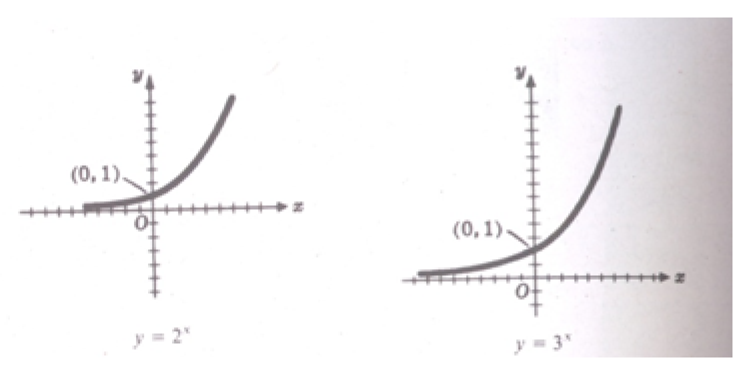
$$t = 8 \text{ days}$$

Exponential Curve

The curve whose equation is $y = a^x$ is called an ***exponential curve***.
The general properties of such curves are that the curve passes through the point $(0,1)$ and that the curve lies above the x axis and x axis as an asymptote of the curve.

Example 3: Sketch the graphs of $y = 2^x, y = 3^x$

The graphs are shown below.



The exponential equation appears frequently in the form $y = ce^{kx}$ where c and k are non zero constants and $e = 2.71828 \dots$. We can solve equations that contain an variable as a exponent if we can convert both sides of the equation to an expression with the same base.

Example 4: Solve the equation $8^x = \frac{1}{2}$

Solution:

$8^x = \frac{1}{2} \Rightarrow (2^3)^x = 2^{-1} \Rightarrow 2^{3x} = 2^{-1} \Rightarrow 3x = -1 \Rightarrow x = -\frac{1}{3}$

Compound Interest If P rupees are deposited in an account with an annual interest rate r, compounded n times per year, then the amount of money in the account after t years is given by the exponential equation

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

The number e , like π, is an irrational number. Like π, it can be approximated with a decimal number. Whereas π is approximately 3.1416, e is approximately 2.7183.
The exponential function based on the number e is called the *natural exponential function*.
One common application of natural exponential functions is with interest bearing accounts. The formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

gives the amount of money in an account if P rupees are deposited for t years at annual interest rate r, compounded n times per year. If we let the number of compounding periods become indefinitely large (that is, we compound the interest every moment), we have an account with an interest that is compounded continuously. The amount of money in the account after t years is given by

$$A(t) = \lim_{n \to \infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= Pe^{rt}$$

Example 5:

Suppose you deposit Rs 500 in an account with an annual interest rate of 8% compounded continuously. Then find the amount of money in the account after 5 years.

Solution

Since the interest is compounded continuously , we use the formula
The interest is compounded continuously. Therefore we use the formula $A(t) = Pe^{rt}$

$$A(t) = 500e^{0.08t}$$

After 5 years, this account will contain

$$A(5) = 500e^{0.08(5)} = 500e^{0.4} = 745.91$$

Example 6:

Suppose you deposit Rs 500 in an account with an annual interest rate of 8% compounded **monthly**. Find the amount of money in the account after 5 years?

Solution

We have,

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where $P = Rs.500, r = 0.08, n = 12, t = 5$

$$A(5) = 500\left(1 + \frac{8}{12 \times 100}\right)^{12 \times 5} = 500\left(\frac{151}{150}\right)^{60} = 744.92$$

Remark: The effect of continuous compounding as compared with monthly compounding is an addition of Rs 0.99.

1.6

Logarithmic Functions

Learning objectives:

- About the logarithmic functions, common Logarithms, Natural Logarithms, Binary Logarithms and Logarithmic Equations.

- The relationship between the Exponential and Logarithmic functions.
  Solve the problems related to the above concepts.

Logarithmic Functions

Logarithms are related to exponents as follows.
Let b be a positive number and $b \neq 1$ then the logarithm of any positive number x to the base b, written $\log_b x$ represents the exponent to which b must be raised to obtain x. That is,

$$\text{If } \ y = \log_b x \quad \text{then} \quad x = b^y$$

Accordingly,

$$\log_2 8 = 3 \qquad \text{since} \qquad 2^3 = 8$$
$$\log_2 64 = 6 \qquad \text{since} \qquad 2^6 = 64$$
$$\log_{10} 100 = 2 \qquad \text{since} \qquad 10^2 = 100$$
$$\log_{10} 0.001 = -3 \qquad \text{since} \qquad 10^{-3} = 0.001$$

Furthermore, for any base b,

$$\log_b 1 = 0 \quad \text{since} \quad b^0 = 1$$
$$\log_b b = 1 \quad \text{since} \quad b^1 = b$$

The logarithm of a negative number and the logarithm of 0 are not defined.
Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, one obtains
$\log_{10} 300 = 2.4771 \quad \log_e 40 = 3.6889 \qquad (e = 2.718281\cdots)$
as approximate answers.

The integral part of the logarithm is called the *characteristic*. The fractional part is called the *mantissa*.

Thus 2 and 3 above are the characteristic, while 0.4771 and 0.6889 are the mantissa of the logarithms of the corresponding numbers.

Three classes of logarithms are of special importance: logarithms to base 10, called ***common logarithms***; logarithms to base e, called ***natural logarithms***; and logarithms to base 2, called ***binary logarithms***.

In the initial mathematical work, it is common to use $\log x$ to mean
$\log_{10} x$ and $\ln x$ to mean $\log_e x$.
In the advanced mathematical work, the term $\log x$ is used for $\log_e x$.

There are two special identities each of which is a consequence of the definition of a logarithm:

$$b^{\log_b x} = x \qquad \text{and} \quad \log_b b^x = x$$

The first identity simply says that we take logarithm of x first and then exponent ate; whereas the second identity says that we take exponential first and then the logarithm. Evidently, both should yield x since *exponential and logarithms are inverse to each other*. However, they can also be formally proved.

1.7

Exponential Equations

Learning objectives:

- To define the exponential equations.
- To solve the exponential equations using logarithms.
  AND
- To practice the related problems.

Exponential Equations

For items involved in exponential growth, the time it takes for a quantity to double is called the ***doubling time***.

For example, if you invest Rs.5000 in an account that pays 5% annual interest, compounded quarterly, you may want to know how long it will take for your money to double in value. You can find this doubling time if you can solve the equation

$$10{,}000 = 5000(1.0125)^{4t}$$

The method of converting both sides of the equation to an expression with the same base may not work for this problem. Logarithms are very important in solving equations in which the variable appears as an exponent.

The above equation is an example of one such equation. Equations of this form are called exponential equations.

An *exponential equation* is an equation where a variable appears in one or more exponents.

If $5^x = 12$ then, so are their logarithms. Notice that both sides of the equation cannot be written as a power of the same base. Now a method in which we take the logarithm on both sides may work.

Example 1: Solve the equation. $5^n = 20$

Solution: Taking the logarithms on both sides of the equation, we have

$$\log 5^n = \log 20 \Rightarrow n \log 5 = \log 20$$

$$\Rightarrow n = \frac{\log 20}{\log 5} = \frac{1.3010}{0.6990} = 1.861$$

Example 2: How long does it take for Rs 5000 to double if it is deposited in an account that yields 5% interest compounded once a year?

Solution

$$10{,}000 = 5000(1 + 0.05)^t \Longrightarrow 2 = (1.05)^t$$

We solve by taking the logarithm of both sides.

$$\log 2 = log(1.05)^t$$
$$= t \log 1.05$$

Dividing both sides by, $\log\ 1.05$ we have

$$t = \frac{\log 2}{\log 1.05} = 14.2$$

It takes a little over 14 years to double if it earns 5% interest per year, compounded once a year.